More examples of using substitution to find integrals.

① $\displaystyle\int_{p=0}^{1} p\sqrt{p^2+3}\,dp = ?$

$du = u'(p)\,dp$

Let $u = p^2+3 \Rightarrow du = 2p\,dp$

$\Rightarrow \frac{1}{2}du = p\,dp.$

$u=p^2+3$
$p=1$
$u=1^2+3=4$

$$= \int_{u=3}^{u=4} \sqrt{u}\,\frac{1}{2}\,du$$

$u=p^2+3$
$u=0^2+3=3$

$$= \frac{1}{2}\int_{u=3}^{4} \sqrt{u}\,du$$

$\int u^{1/2}\,du = \frac{u^{3/2}}{3/2}+C$

$= \frac{2u^{3/2}}{3}+C$

$$= \frac{1}{\cancel{2}}\cdot\frac{\cancel{2}u^{3/2}}{3}\Big|_3^4$$

$$= \left(\frac{4^{3/2}}{3}\right) - \left(\frac{3^{3/2}}{3}\right) = \boxed{\frac{8-3^{3/2}}{3}}.$$

$4^{3/2} = (4^{1/2})^3 = 2^3 = 8$

(2) $\int \cot(5y)\,dy$

$= \int \frac{\cos(5y)}{\sin(5y)}\,dy$

$u = \cos(5y) \Rightarrow du = -5\sin(5y)\,dy$

$= \int \underline{\qquad}\,du$

we need this in the top, but it's not there.

Good try

Let $u = \sin(5y)$

$\Rightarrow du = 5\cos(5y)\,dy$

$\frac{1}{5}du = \cos(5y)\,dy$

$\therefore \int \frac{\cos(5y)\,dy}{\sin(5y)} = \int \frac{\frac{1}{5}du}{u}$

$= \frac{1}{5}\int \frac{du}{u} = \frac{1}{5}\int \frac{1}{u}\,du$

$= \frac{1}{5}\ln|u| + C$

$= \boxed{\frac{1}{5}\ln|\sin(5y)| + C}$

Check: $\left[\frac{1}{5}\ln(\sin(5y))+C\right]'$

$$= \frac{1}{5}\frac{1}{\sin(5y)}\cdot 5\cos(5y) = \frac{\cos(5y)}{\sin(5y)} = \boxed{\cot(5y)}.$$

③ $$\int_{x=-1}^{1} \frac{e^x}{1+e^{2x}}\,dx = ?$$

Let $u = 1+e^{2x} \Rightarrow du = 2e^{2x}\,dx$

Great try!

don't have that.

Let $u = e^x \Rightarrow \boxed{du = e^x\,dx}$

$\Rightarrow 1+e^{2x} = 1+(e^x)^2 = 1+u^2$

$$= \int_{u=e^{-1}}^{u=e^{1}} \frac{1}{1+u^2}\,du \qquad \frac{du}{1+u^2}$$

$$= \arctan(u)\Big|_{e^{-1}=\frac{1}{e}}^{e^{1}=e}$$

$$= \boxed{\arctan(e) - \arctan\left(\tfrac{1}{e}\right)}$$

4 Find $\displaystyle\int_{-1}^{1} \frac{e^{2x}}{1+e^{2x}}\,dx$.

Let $u = 1+e^{2x}$ $\quad du = 2e^{2x}dx$

$\frac{1}{2}du = e^{2x}dx$

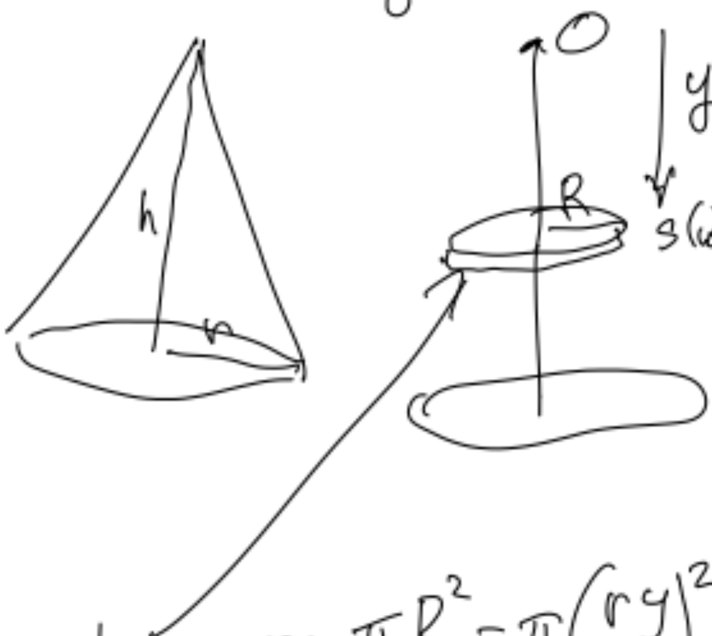
$$= \int_{u=1+e^{-2}}^{u=1+e^{2}} \frac{\frac{1}{2}du}{u} = \frac{1}{2}\int_{1+e^{-2}}^{1+e^{2}} \frac{1}{u}\,du$$

$$= \frac{1}{2}\ln|u|\Big|_{1+e^{-2}}^{1+e^{2}}$$

$$= \left(\frac{1}{2}\ln(1+e^{2})\right) - \left(\frac{1}{2}\ln(1+e^{-2})\right)$$

$$= \boxed{\ln\left(\sqrt{\frac{1+e^{2}}{1+e^{-2}}}\right)}$$

(5) Application: Find the formula for the volume of a cone:



slice at position y

$$\frac{h}{r} = \frac{y}{R}$$
$$Rh = ry$$
$$R = \frac{ry}{h}$$

slice area $\pi R^2 = \pi\left(\frac{ry}{h}\right)^2 = \frac{\pi r^2 y^2}{h^2}$

$$\text{Whole Volume} = \int_0^h \underbrace{\left(\frac{\pi r^2}{h^2} y^2\right)}_{\text{area}} \underbrace{dy}_{\text{thickness}}$$

$$= \frac{\pi r^2}{h^2}\int_0^h y^2\,dy = \frac{\pi r^2}{h^2}\left(\frac{y^3}{3}\Big|_0^h\right)$$

$$= \frac{\pi r^2}{\cancel{h^2}}\frac{h^{\cancel{3}\,1}}{3} - \frac{\pi r^2 0^2}{h^2 3}$$

$$= \boxed{\frac{\pi r^2 h}{3}}$$ Volume of a cone.